Mechanics of Solids

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§1.1 Introduction

Mechanics is the analysis of forces on bodies. This encompasses statics and dynamics, which concerns forces on rigid bodies. Mechanics also encompasses solids and fluids, which which are classified as deformable bodies that change shape. We study this for the following two reasons:

1. **Design** - We want to make efficient use of materials.
2. **Failure Safety** - We do not want the materials to break (**strength**), or to deform (change of shape).

Objects exhibit behaviour dependent on their structure and composition.

1. **Structural Behaviour** concerns forces, moments, boundary conditions, and reaction forces.
2. **Material Behaviour** concerns the **stresses** and **strains** on materials. Stresses and strains are related by material properties.

§1.2 Internal Reactions

Recall that we can divide a body into sections. When doing so, the normal force is directed away from the split sections, while the shear force is directed along the split. There is also a moment for both sections. These three internal reactions are equal in magnitude and opposite in direction. In this course, we will deal with the following internal reactions:

- N is the **axial** or **normal force** that is normal to the surface.
- V is the **shear force** that is parallel to the surface.
- M is the **bending moment** used for calculations involving beams with transverse bending.
- T is the **axial torque** used for calculations involving shafts and twisting.

Calculations in this course must be performed on bodies in equilibrium. Recall that we need to satisfy

$$\sum F = 0,$$
$$\sum M = 0.$$

Shear and Bending Moment Diagrams are graphs of internal reactions. These are usually drawn for the reactions along a beam. After calculating from equilibrium the unknown forces, we section along a point of our choosing. Distributed loads are split along the section. By convention, a split along a point C will produce an axial force N_C to the right, a shear force V_C downwards, and a bending moment M_C counterclockwise for the left section. The right section would have these values in the opposite direction.

Remark 1.1. It is usually advantageous to take the sum of the moments at the section when solving for the internal reactions of a section.

§1.3 Shear and Bending Moment Diagrams

When a load is applied or a distributed load changes, we split just to the left and just to the right. This is associated with an unknown N, V, and M. We will express V and M as a function of x along the beam. We now solve in terms of x. To solve these problems, we apply the following steps:

1. Draw the free body diagram with reactions.
2. Ensure equilibrium by determining unknown variables.
3. Solve by splitting into sections. Consider either the left or right section.
4. After splitting, we now need to consider the shear and bending moment.
5. Determine the shear and bending moment in terms of x.
6. Repeat for the remaining sections.
7. Plot diagrams noting zeroes, maximums, minimums, and points of interest (where forces were applied).

Remark 1.2. The derivative $\mathrm{d}M/\mathrm{d}x = V$. Thus, the derivative of bending moment diagram results in the shear diagram.

§2 July 7, 2017

§2.1 Relationship Between Shear and Bending Moment

We can relate the distributed load $w(x)$ with its effect on V and M. Consider the differential element $\mathrm{d}x$. On either side of the differential element under a distributed load, we require the resulting reaction shear and moment to establish equilibrium. Then, by considering the differential change in shear, we find that

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -w.$$

A similar approach can be applied to the bending moment. Doing so, we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V,$$

$$\Delta M = M_D - M_C = \int_C^D V \mathrm{d}x.$$

Note that when $\mathrm{d}M/\mathrm{d}x = 0$, $V = 0$. Thus, $V = 0$ when we have a maximum or minimum M. The first equation above is used to determine the shape of the shear diagram. The second equation above is used to find the shape of the bending moment diagram, while the third equation above is used to calculate the value of M at different points of interest.

We can also relate the effect of a concentrated load F with its effect on V. Once again considering the differential element, we find that

$$\mathrm{d}V = -F.$$

Thus, for a downward F, the shear steps downward by a magnitude of F, and for an upward F, the shear steps upward by a magnitude of F. The initial and final shear either stepping upwards or downwards is given by the force at that location.

The effect of an initial bending moment M_0 gives us the relation

$$\mathrm{d}M = -M_0.$$

Thus, for a counterclockwise M_0, the change in the moment will be negative and steps downward with a magnitude of M_0. Similarly, a clockwise M_0 results in a positive change in moment and steps upward with a magnitude of M_0. The initial and final moment either stepping upwards or downwards are given by the external moment at that location.

§3 July 10, 2017

§3.1 Axial Stress

Consider a long slender bar with length L and cross sectional area A. **Axial stress**, denoted by σ, is defined as the force divided by the cross sectional area. The strength of a bar depends on the force applied, the length of the bar, the cross sectional area, and the properties of the material.

Consider a cylindrical bar with a concentrated force F applied at one end. To ensure equilibrium, the total force on the other end must also be F. On this end, the force is distributed over the face. Since stress is the force per unit area, we can relate average stress by

$$\sigma_{avg} = \frac{F}{A},$$

where F is the force, and A is the cross sectional area. The units of stress are in Pascals.

However, stress changes from point to point. Thus, for an area ΔA with a force of ΔF, we have

$$\sigma_{point} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}.$$

By conventional, we will consider a tensile force to be positive, and a compressive force to be negative.

A concentrated force applied to the ends of a long cylindrical bar is intuitively concentrated at the centroid. At the ends, σ is not uniform. Near the middle region, σ is very nearly uniform. The region of the bar where we can assume uniformity is referred to as the uniform region of stress. This follows from **Saint-Venant's Principle**, which states that the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load. In this course, we will generally make this assumption when problem solving. Considering the differential elements of $\mathrm{d}A$ and $\mathrm{d}F$, we find that

$$F = \int \mathrm{d}F = \int_A \sigma \mathrm{d}A \approx \sigma A,$$

where σ is the average stress in the middle uniform region.

Given two rods of cross sectional area A_1 and A_2, each with the same force F applied at both ends, then their stresses are given by

$$\sigma_1 = \frac{F}{A_1},$$

$$\sigma_2 = \frac{F}{A_2}.$$

We note that if $A_2 > A_1$, then $\sigma_2 < \sigma_1$. A material breaks or fails when it reaches an ultimate stress σ_u. Thus, we would expect a bar with a larger cross sectional area to break after a similar bar with a smaller cross sectional area.

§3.2 Axial Stress Problems

Suppose we are given the maximum stress beyond which a bar will break. We are then given a diagram of a bar in equilibrium, with the axial forces to the left balancing the axial forces to the right. These forces are applied at different locations along the bar. We would like to determine whether the bar fails. To solve these problems, we apply the following steps:

1. Section along the bar whenever a new axial force appears.
2. We can then draw an **axial force diagram**. This is done by plotting the force that acts along each section of the bar. These often take the form of horizontal steps.
3. Calculate the stress σ of each region by dividing the force on that section by the cross sectional area of that section.
4. Plot the **stress diagram** across the entire bar.
5. Compare with the maximum allowable stress.

§3.3 Rivets, Bolts, and Pins

Consider a connector holding two bars together. A force of F is applied to one bar, so a similar force in the opposite direction is applied to the other bar. In the connector, the top half is sheared one way, and the bottom half is sheared the other way. We section in between the two forces, to find that the shear V is equal to the force F. **Shear stress**, denoted by τ, is given by

$$\tau_{avg} = \frac{(F = V)}{A}.$$

Bearing stress occurs in the material when a connector applies a force to the bar. There is non-uniform compressive stress from the pin. This is always compressive, so F is always negative. For a bar of thickness t, along which a force F is applied along the connector of diameter d, the bearing stress is given by

$$\sigma_{bearing} \approx \frac{F}{t \cdot d}.$$

§3.4 Stresses on an Oblique Plane

Instead of sectioning vertically, we can section along θ from the vertical. Doing so, the internal axial force F remains the same. However, we now have a shear force V along the oblique section, and a normal force N perpendicular to this oblique section, where θ is the angle from the internal axial force F to N. We note that

$$N = P \cos\theta,$$

$$V = P \sin\theta.$$

Additionally, $A_\theta > A_0$, where A_θ is the oblique area, and A_0 is the original cross sectional area. This results since we are not splitting vertically. We have

$$A_0 = A_\theta \cos\theta,$$

$$A_\theta = \frac{A_0}{\cos\theta}.$$

Now, we have different formulas for axial and shear stress:

$$\sigma = \frac{N}{A_0} = \frac{F\cos\theta}{A_0/\cos\theta} = \frac{F\cos^2\theta}{A_0},$$

$$\tau = \frac{V}{A_0} = \frac{F\sin\theta}{A_0/\cos\theta} = F\cos\theta\sin\theta A_0.$$

There are several different scenarios of interest, as stress depends on orientation:

1. If $\theta = 0$, then $\sigma = F/A_0$ and $\tau = 0$. This is the maximum value of σ.
2. If $\theta = 45$, then $\sigma = F/2A_0$ and $\tau = F/2A_0$. This is the maximum value of τ.
3. If $\theta = 90$, then $\sigma = 0$ and $\tau = 0$.

If we plot σ or τ against θ where $0 \leq \theta \leq 90$, this can be visualized as smooth curves that indicate the axial and shear stress in different orientations.

§3.5 Factor of Safety

We recall that **ultimate stress** is given by

$$\sigma_u = \frac{F_u}{A}.$$

In the above equation, σ_u is the ultimate stress before material failure. This is determined from experiment, and is unique to the material. Thus, F_u is the **ultimate load**. The allowable stress and allowable load are necessarily less than the ultimate stress and ultimate load respectively. **Factor of Safety** is defined in design codes, and is given by

$$FS = \frac{F_u}{F_A} > 1,$$

where FS is the factor of safety, F_u is the ultimate load, and F_A is the allowable load. With a linear relationship between F_u and σ_u, we sometimes calculate FS using stress instead of loads.

§3.6 Axial Strain

A bar of length L under a force F may stretch or deform by a length of δ. **Axial strain**, denoted by ϵ, is the change in length divided by the original length. Thus,

$$\epsilon = \frac{\delta}{L}.$$

This is also known as **engineering strain**, and is a dimensionless quantity. In this course, this value will be a small number in most applications. By convention, δ is positive if there is an increase in length associated with tension, and δ is negative if there is a decrease in length associated with compression.

In typical materials, if we plot F against σ, we first encounter a linear region, followed by two bumps. This **load displacement curve** depends on A and L. This illustrates structural behaviour. Normalizing the values, we can then plot a **stress-strain curve**. This curve has the same general shape of the load displacement curve, but is now independent of A and L. The stress-strain curve is now unique for a given material, and thus illustrates material behaviour. The linear slope obtained from dividing $\Delta\sigma$ by $\Delta\epsilon$ is known as **Young's Modulus**, or the **Modulus of Elasticity**. This is given by E, and is related by **Hooke's Law**,

$$\sigma = E\epsilon.$$

The region with linear slope is known as the **elastic region**, the remaining region is known as the **plastic region**, the **yield strength** is given by the σ_y at which we transition from the elastic region to the plastic region (beyond which we have permanent deformation), and σ_u is the maximum σ that occurs at the second bump.

Example 3.1

Aluminum has a plastic region that is often arbitrarily defined at say a 0.2% offset. Brittle materials such as cast iron and concrete break after briefly leaving the linear region.

§4 July 12, 2017

§4.1 Stress Examples

To solve stress problems, we use the following steps:

1. We first need to solve for equilibrium. We will write all of the reaction forces required for equilibrium in terms of the load w.

2. We now need to consider all of the maximum allowable σ and τ.

3. First consider failure in a bars. For a given σ_{allow}, we equate this with the force on the bar (written in terms of w) divided by the cross sectional area of the bar. Solving this, we obtain w_{max} in N/m. This is the max load for w for a tensile or compressive load in the bar.

4. Note that we can also consider the tensile stress from the bar. Reinforced bars would have a certain width and height associated with them. There are four sections of this, so we equate σ with the force on the bar divided by $(w * h * 4)$, where w is the width and h is the height.

5. Now, consider failure in pins. We generally have **double shear**, where sectioning gives us $V_{pin} = F/2$. The shear in the pin is therefore half of the force applied to the pin (written in terms of w). For a given τ_{allow}, we equate this with V_{pin} divided by the cross sectional area of the pin. Solving this, we obtain w_{max} in N/m. This is the max load for w in the pin.

6. Note that for pins for which we have separated into vertical and horizontal components, we must first consolidate the force by taking the net force at that location. This may involve finding $F_A = \sqrt{A_x^2 + A_y^2}$.

7. The maximum load that can be placed on the system will therefore be the lowest calculated w from all the pins and bars.

8. The bearing stress can be calculated by dividing the shear ($V_{pin} = F/2$) on the bar by the width (same width as above) multiplied with the diameter of the pin.

9. To determine factor of safety, we determine F_u from σ_u and the cross sectional area. We then determine the ultimate load P_u from relating it to F_u in the equilibrium reaction equations. Comparing this with P_{allow}, we can determine the factor of safety.

§4.2 Axial Deformation

We recall that $\sigma = E\epsilon$. We can substitute from the definitions of stress and strain and rearrange to obtain

$$\delta = \frac{FL}{AE},$$

where F is the load at the two ends of the bar, L is the length of the bar, A is the cross sectional area of the bar, and E is Young's Modulus for the material. By the principle of superposition, we can simply add changes in length for different components of a bar, so

$$\delta_{total} = \sum \frac{F_i L_i}{E_i A_i}.$$

§4.3 Axial Statically Indeterminate Problems

We can determine the change in position of a point in a bar by considering the length of the bar to be from the fixed end to the point of interest. Thus, δ now indicates the change in position of this point. However, we cannot determine the change in length of the point of interest when both ends of the bar are fixed. To solve these problems, we need to generate another equation from the constraints. In this case, we have the additional constraint that

$$\delta_{total} = \delta_{AC} + \delta_{CB} = 0.$$

Thus, we section along the initial application of the force F after determining the equilibrium condition of $F_A + F_B = F$. Take note to draw δ in the same direction as its corresponding force F_A or F_B. We obtain

$$\sigma_{AC} = \frac{F_A L_{AC}}{A_{AC} E},$$

$$\sigma_{CB} = -\frac{F_B L_{CB}}{A_{CB} E},$$

where the negative is introduced as one side is in compression. Combining the three equations, we obtain

$$F_A L_{AC} = F_B L_{CB}.$$

To summarize, we use the following steps for axial statically indeterminate problems:

1. Determine the equilibrium equations from statics.
2. Determine a compatibility or geometric constraint.
3. Link the forces to displacement through load displacement.

§5 July 14, 2017

§5.1 Superposition (Flexibility or Force Method)

A more generalized technique for solving statically indeterminate problems is to first remove a redundant support to obtain a statically determinate problem. We apply external loads only. Doing this, we are able to determine the change in length δ_B. We then apply the force F_B to push the bar back so that the total change in length is 0. This is done without any external loads acting on the system. We then consider the geometric constraints. This may be rotation due to the location of the force.

§5.2 Temperature Change

We note that length change also depends on temperature. We denote the **coefficient of thermal expansion** with α, where

$$\delta_T = \alpha(\Delta T)L.$$

Note that we are concerned with a change in temperature, where an increase in temperature is associated with an increase in length, and a decrease in temperature is associated with a decrease in length. There is no stress when the bar is unconfined. When we have a confined bar, an increase in temperature is associated with compressive forces, and a decrease in temperature is associated with tension forces. To solve these problems, we remove redundant constraints, apply ΔT to find the new length, then apply the reaction force only. Doing this, we find that

$$\alpha(\Delta T) = \frac{F}{EA},$$

$$\sigma = -\frac{F}{A} = -E\alpha(\Delta T).$$

where α is the coefficient of thermal expansion, ΔT is the change in temperature, F is the reaction force, E is Young's Modulus, and A is the cross sectional area.

§6 July 17, 2017

§6.1 Indeterminate Temperature Problems

For bars in series, they experience the same force under constrained thermal expansion. The total increase in length is equal to the sum of the increase in length of each component of the bar. That is, they experience the same force but expand to different lengths. For bars in parallel, they experience different forces under constrained thermal expansion. The increase in length is equal among all components of the bar. That is, they experience the same expansion in length, but different forces. To solve these problems, we use superposition to consider the temperature effects only, then consider the reaction force effects only.